

Manifestly Supersymmetric Effective Lagrangians on BPS Solitons

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Abstract

A systematic method to obtain the effective Lagrangian on the BPS background in supersymmetric gauge theories is worked out, taking domain walls and vortices as concrete examples. The Lagrangian in terms of the superfields with four preserved supercharges is expanded in powers of the slow-movement parameter λ . The expansion gives the superfield form of the BPS equations at $\mathcal{O}(\lambda^0)$, and all the fluctuation fields at $\mathcal{O}(\lambda^1)$. The density of the Kähler potential for the effective Lagrangian follows as an automatic consequence of the λ expansion making (four preserved) supercharges manifest.

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1 Introduction

Various topological solitons have been considered in constructing models in the brane-world scenario [1]–[3]. It has been quite useful to consider supersymmetric gauge theories for phenomenological purposes [4]. The simplest solitons are domain walls giving one extra dimension, which have been studied extensively in theories with four supercharges [5]–[8] and in theories with eight supercharges [9]–[23]. The next simplest solitons are vortices [24]–[39] giving two extra dimensions. Monopoles and Yang-Mills instantons are solitons with three and four extra dimensions, respectively. If a field configuration preserves a part of supersymmetry (SUSY), it satisfies the field equation automatically [40]. Such configuration is called the Bogomol’nyi-Prasad-Sommerfield (BPS) state [41]. We usually obtain a family of BPS solutions characterized by parameters, called moduli. These moduli parameters constitute the moduli spaces. Although solutions and their moduli spaces were established and discussed extensively for instantons [42, 43] and monopoles [44, 43], those for domain walls and vortices have been much less studied for a long time. Recently the moduli space of BPS domain walls in a supersymmetric non-Abelian gauge theory with eight supercharges has been completely characterized and explicit solutions have been found for strong gauge coupling with complete moduli and for finite coupling with partial moduli [14, 15]. This is achieved by solving the hypermultiplet BPS equation and by rewriting the remaining BPS equation into a “master equation” for a gauge invariant quantity Ω . The moduli space of vortices is also characterized in Abelian gauge theory [26]–[28]. Vortices in non-Abelian gauge theory, called non-Abelian vortices, have been recently found [29]–[35], and their moduli space has been determined [29, 34]. These solitons in the Higgs phase are extensively reviewed recently [45]. It is important to construct the low-energy effective Lagrangian of the localized modes on such solitons for brane-world scenario. In order to obtain the low-energy effective Lagrangian, the standard method is to promote the moduli parameters of the background soliton into fields on the world volume of the soliton [46]. The method is based on the assumption of the weak dependence on the world-volume coordinates, and gives the low-energy effective Lagrangian which contains all nonlinear terms with two derivatives or less. Another interesting aspect of the soliton dynamics is the scattering of solitons [46, 47]: it has been extensively studied primarily for cases without the spacial world volume.

The purpose of our paper is to establish a systematic method to obtain the effective Lagrangian on BPS background in supersymmetric gauge theories maintaining the preserved SUSY manifest. We explicitly work out our method by taking domain walls and vortices in the supersymmetric $U(N_C)$ gauge theories with eight supercharges with $N_F(\geq N_C)$ hypermultiplets in the fundamental representation as illustrative examples. Although we work in the space-time dimensions highest allowed by supersymmetry, namely vortices and walls in six and five dimensions, respectively, our discussion is applicable completely in lower dimensions which can be obtained simply by dimensional reductions. Since we can naturally specify the order of magnitude in powers of the slow-movement parameter λ for various fields, we obtain a systematic expansion of the Lagrangian in powers of λ . Then a superfield form of the BPS equations results at the zero-th order in λ , and the superfield equation to determine all the fluctuation fields follows at the next order. Here we have retained up to the terms of order λ^2 in the Lagrangian, in order to obtain the effective Lagrangian at the lowest nontrivial order, namely up to two derivatives. We anticipate that retaining higher powers of λ in our systematic expansion will offer a systematic method to compute the effective Lagrangian with higher derivative terms. Since four SUSY are preserved manifest throughout our procedure, our result is summarized as a density of the

Kähler potential in four SUSY superspace. By integrating over the extra dimensions, the Kähler potential of the effective Lagrangian is obtained. Our results should be useful to study soliton scattering in $U(N_C)$ gauge theories. We have also worked out explicitly the effective Lagrangian for multi-wall systems. For finite gauge coupling, we have illustrated the use of our effective Lagrangian by solving the 1/2 BPS lumps in the double-wall effective Lagrangian and found that the energy of boojum [37], [38] is correctly reproduced.

A number of formulations of theories with eight supercharges in space-time dimensions greater than four have been devised to use superfields maintaining only the four supercharges manifest [48]–[54]. We have succeeded to obtain a manifestly supersymmetric method to determine the low-energy effective Lagrangian using the superfields for four preserved SUSY. Since we are interested in topological solitons whose energies are generally given by topological charges, it is necessary to keep track of all the total derivative terms, which are often neglected in using the superfield formalism for four SUSY. Moreover, it is important to realize that the auxiliary fields of the superfields for the four SUSY is different from auxiliary fields for the eight SUSY by total derivative terms. From these two facts, we find that the topological charges follow as an automatic consequence of rewriting the fundamental (five or six dimensional) Lagrangian in terms of the superfield for four SUSY, in the cases of domain walls or vortices, respectively. Namely, the Lagrangian in terms of the superfield with four preserved SUSY manifest is different from the fundamental Lagrangian with manifest eight SUSY by a total divergence term which precisely gives the topological charge of the BPS soliton.

To obtain the Kähler potential of the effective Lagrangian, we first solve the BPS equation for hypermultiplets and rewrite the Lagrangian in terms of the remaining dynamical degree of freedom, the gauge invariant quantity Ω . The result is given by the Kähler potential density $K(\Omega, y)$ in terms of the gauge invariant quantity Ω as the dynamical variable. If we replace Ω by the solution Ω_{sol} of the master equation and integrate over the extra dimensions, we finally obtain the effective Lagrangian. One interesting feature is that the Kähler potential density $K(\Omega, y)$ in five or six dimensions (before Ω is replaced by Ω_{sol}) serves as a Lagrangian from which the master equation for the gauge invariant quantity Ω can be derived by the usual minimal action principle. Our derivation of the Kähler potential density explains this empirical observation, since the Lagrangian with the Kähler potential density can be understood as the fundamental Lagrangian after the functional integral over the hypermultiplet in this approximation.

In sect.2, after a brief review of component formalism for walls, we express the fundamental Lagrangian in terms of superfields with four SUSY and expand the Lagrangian in five dimensions in powers of the slow-movement parameter λ , to obtain the density of the Kähler potential for the effective Lagrangian on 1/2 BPS walls. In sect.3, a superfield treatment of the effective Lagrangian on vortices is worked out as another example to show the usefulness of our method. In sect.4, we obtain explicitly the effective Lagrangian of multi-walls for exact solutions at strong coupling limit as well as at a discrete finite value of gauge coupling. Using double-wall effective Lagrangian, lumps are worked out to give boojum correctly. Sect.5 is devoted to brief discussion. Appendix A contains a component approach for the wall case. Appendix B contains useful formulas in the six-dimensional Lagrangian for the vortex case.

2 Slow-move Approximation for Walls

2.1 Component Formalism of Slow-move Approximation

Let us here review briefly our model and the usual component method to solve the BPS equations. The bosonic parts of the Lagrangian with a common gauge coupling constant g for $U(N_C)$ in five dimensions is given by

$$\begin{aligned} \mathcal{L}|_{\text{boson}} = & \text{Tr} \left[-\frac{1}{2g^2} F_{MN}(W) F^{MN}(W) - \frac{1}{g^2} (\mathcal{D}_M \Sigma)^2 + \frac{1}{g^2} (Y^a)^2 - c^a Y^a \right. \\ & \left. - \mathcal{D}^M H^i (\mathcal{D}_M H^i)^\dagger + F^i F^{i\dagger} - (\Sigma H^i - H^i M) (\Sigma H^i - H^i M)^\dagger + Y^a (\sigma^a)_{ij} H^j H^{i\dagger} \right]. \end{aligned} \quad (2.1)$$

Here the bosonic components in the vector multiplet are gauge fields W_M , the real scalar fields Σ and the triplet of auxiliary fields Y^a , $a = 1, 2, 3$, all in the adjoint representation, and those in the hypermultiplet are the doublets of the complex scalar fields H^i and the auxiliary fields F^i , $i = 1, 2$ which can be assembled into $N_C \times N_F$ matrices. The indices $M, N = 0, 1, \dots, 4$ run over five-dimensions, and the mostly plus signature is used for the metric $\eta_{MN} = \text{diag.}(-1, +1, \dots, +1)$. The covariant derivatives are defined as $D_M \Sigma = \partial_M \Sigma + i[W_M, \Sigma]$, $D_M H^i = (\partial_M + iW_M) H^i$, and field strength is defined as $F_{MN} = \frac{1}{i}[D_M, D_N] = \partial_M W_N - \partial_N W_M + i[W_M, W_N]$. After eliminating auxiliary fields Y^a , the scalar potential V is given by

$$\begin{aligned} V = & \frac{g^2}{4} \text{Tr} \left[(H^1 H^{1\dagger} - H^2 H^{2\dagger} - c \mathbf{1}_{N_C})^2 + 4 H^2 H^{1\dagger} H^1 H^{2\dagger} \right] \\ & + \text{Tr} [(\Sigma H^i - H^i M) (\Sigma H^i - H^i M)^\dagger], \end{aligned} \quad (2.2)$$

with the hypermultiplet mass matrix $M = \text{diag}(m_1, \dots, m_{N_F})$ and the Fayet-Iliopoulos parameter taken along the third direction in $SU(2)_R$ as $c_a = (0, 0, c)$ with $c > 0$.

By requiring half of SUSY to be preserved, we obtain the 1/2 BPS equations for domain walls which depend on y only

$$\mathcal{D}_y H^1 = -\Sigma H^1 + H^1 M, \quad \mathcal{D}_y H^2 = \Sigma H^2 - H^2 M, \quad (2.3)$$

$$\mathcal{D}_y \Sigma = \frac{g^2}{2} (c \mathbf{1}_{N_C} - H^1 H^{1\dagger} + H^2 H^{2\dagger}), \quad 0 = g^2 H^1 H^{2\dagger}. \quad (2.4)$$

The solution of the BPS equations saturates the BPS bound for the tension of the (multi-)wall

$$\begin{aligned} T_w &= \int_{-\infty}^{+\infty} dy \mathcal{E}_w = \int_{-\infty}^{+\infty} dy \partial_y \left[\text{Tr} [c \Sigma - (\Sigma H^1 H^{1\dagger} - H^1 M H^{1\dagger}) + (\Sigma H^2 H^{2\dagger} - H^2 M H^{2\dagger})] \right] \\ &= c [\text{Tr} \Sigma]_{-\infty}^{+\infty} \end{aligned} \quad (2.5)$$

where the energy density is denoted as \mathcal{E}_w . It has been shown that the hypermultiplet BPS equation (2.3) can be solved by [14], [15]

$$\Sigma + iW_y = S^{-1}(y) \partial_y S(y), \quad W_\mu = 0, \quad (\mu = 0, \dots, 3) \quad (2.6)$$

$$H^1 = S^{-1}(y) H_0 e^{My}, \quad H^2 = 0, \quad (2.7)$$

where the moduli matrix H_0 is obtained as an integration constant carrying all the parameters of the solution, namely moduli. The moduli matrices related by the following V -equivalence transformations are physically equivalent:

$$H_0 \rightarrow VH_0, \quad S(y) \rightarrow VS(y), \quad V \in GL(N_C, \mathbf{C}). \quad (2.8)$$

The vector multiplet BPS equation (2.4) can be converted to the following “master equation” for a gauge invariant quantity $\Omega \equiv SS^\dagger$ [14]

$$\partial_y (\Omega^{-1} \partial_y \Omega) = g^2 c (\mathbf{1}_{N_C} - \Omega^{-1} \Omega_0), \quad \Omega_0 \equiv c^{-1} H_0 e^{2My} H_0^\dagger. \quad (2.9)$$

The matrix function S can be determined from the solution Ω of this master equation by fixing a gauge, and all the other fields can be obtained from S and H_0 . Since the BPS soliton has co-dimension one, the solution represents (multiple) domain walls. There are two characteristic mass scales in this system: mass differences Δm of hypermultiplets, and the mass scale in front of the master equation $g\sqrt{c}$. In the strong coupling limit $g\sqrt{c} \gg \Delta m$, the vector multiplet serves to give constraints to hypermultiplets leading to the nonlinear sigma model [55], whose BPS domain wall solutions have been obtained exactly [14], [15].

The low-energy effective Lagrangian on the world volume of solitons is given by promoting the moduli parameters to fields on the soliton and by assuming the weak dependence on the world-volume coordinates of the soliton [46]. In the case of domain walls, all the moduli parameters are contained in the moduli matrix H_0 and constitute the complex Grassmann manifold [14], [15]. Denoting the inhomogeneous coordinates of the complex Grassmann manifold in the moduli matrix H_0 as ϕ^α , we promote them to fields which depend on x^μ , ($\mu = 0, 1, 2, 3$)

$$H_0(\phi^\alpha) \rightarrow H_0(\phi^\alpha(x)), \quad (2.10)$$

which has also been studied in our recent review article [45]. We introduce “the slow-movement parameter” λ , which is assumed to be much smaller than the typical mass scale in the problem, in our case, Δm and $g\sqrt{c}$.

$$\lambda \ll \min(\Delta m, g\sqrt{c}). \quad (2.11)$$

The nonvanishing fields of the 1/2 BPS background have contributions independent of λ , and derivatives in terms of the world volume coordinates are assumed to be of order λ , expressing the weak dependence on the world-volume coordinates

$$H^1 \sim \mathcal{O}(1), \quad \Sigma \sim \mathcal{O}(1), \quad \partial_\mu \sim \mathcal{O}(\lambda). \quad (2.12)$$

Those fields which vanish in the background solution can now have nonvanishing values, induced by the fluctuations of the moduli fields of order λ

$$W_\mu \sim \mathcal{O}(\lambda), \quad H^2 \sim \mathcal{O}(\lambda), \quad (2.13)$$

$$\mathcal{D}_\mu H^1 \sim \mathcal{O}(\lambda), \quad \mathcal{D}_\mu \Sigma \sim \mathcal{O}(\lambda), \quad F_{\mu y}(W) \sim \mathcal{O}(\lambda), \quad (2.14)$$

and other components of the field strength are higher orders in λ . If we decompose the field equations in powers of λ , we find that order λ^0 equations are automatically satisfied by the BPS equations (2.3) and (2.4). However, it becomes more and more complicated to solve the field

equation at higher orders in the expansion in powers of λ , since various fields that vanish in the background become nonvanishing, and need to be solved. For instance the equation of motion for the gauge field fluctuations W_μ reads

$$0 = \frac{1}{g^2} \mathcal{D}_y F_{\mu y} + \frac{i}{g^2} [\Sigma, \mathcal{D}_\mu \Sigma] + \frac{i}{2} (H^\dagger \mathcal{D}_\mu H - \mathcal{D}_\mu H^\dagger H), \quad (2.15)$$

as given in Appendix A. To obtain the solution of this equation, one needs to do a long and tedious calculation, leading finally to Eq.(A.2) in Appendix A. Further long calculations give the Kähler metric and Kähler potential for the effective Lagrangian in Eqs.(A.8) and (A.9) in Appendix A. The basic reason for these complications is that component fields are used to expand in powers of λ without exploiting the constraint of SUSY. We leave a brief outline of this procedure in terms of component fields to Appendix A, in order to facilitate a comparison to our result in terms of superfields. We shall show in the next section that maintaining the preserved SUSY manifest greatly helps to determine these newly nonvanishing fields and to organize the expansion of field equations in powers of λ .

2.2 Superfield Formalism of Slow-move Approximation

The BPS wall background conserves a half of SUSY. Thus an action for fluctuations around the BPS background can be written in term of superfield respecting the surviving half of SUSY.

Let us define the superfields¹ using two component spinor Grassmann coordinates $\theta^\alpha, \theta_{\dot{\alpha}}$. The components of superfields are fields in five dimensions. A vector multiplet with eight SUSY consists of a real vector superfield $\mathbf{V}(= \mathbf{V}^\dagger)$ and an adjoint chiral superfield Φ ($\bar{D}_{\dot{\alpha}} \Phi = 0$) in terms of superfield with four supercharges. The vector superfield \mathbf{V} contains a gauge field $W_\mu, \mu = 0, \dots, 3$ for the four spacetime dimensions, the half of gaugino field λ_+ , and an auxiliary field \mathcal{Y}^3 . If one takes the Wess-Zumino gauge, it becomes explicitly as

$$\mathbf{V}|_{\text{WZ}} = -\theta \sigma^\mu \bar{\theta} W_\mu + i\theta^2 \bar{\theta} \bar{\lambda}_+ - i\bar{\theta}^2 \theta \lambda_+ + \frac{1}{2} \theta^2 \bar{\theta}^2 \mathcal{Y}^3, \quad \mathcal{Y}^3 \equiv Y^3 - \mathcal{D}_y \Sigma, \quad (2.16)$$

where the auxiliary field \mathcal{Y}^3 of the superfield for four SUSY is shifted from the auxiliary field Y^3 for eight SUSY by the covariant derivative of adjoint scalar Σ along the fifth coordinate (the extra dimensions) y [53], [54]. This difference becomes important in identifying the topological charge later. The chiral scalar superfield Φ contains a complex scalar field made of the adjoint scalar Σ and the fifth component of the gauge field W_y as the real and imaginary part respectively, and the other half of gaugino λ_- and a complex auxiliary field $Y^1 + iY^2$

$$\Phi = \Sigma + iW_y + \sqrt{2}\theta(-i\sqrt{2}\lambda_-) + \theta^2(Y^1 + iY^2). \quad (2.17)$$

The hypermultiplets are represented by a chiral superfields \mathbf{H}^1 and an anti-chiral superfield \mathbf{H}^2 . The (anti-) chiral superfield \mathbf{H}^1 (\mathbf{H}^2) consists of the physical complex scalar field H^1 (H^2), hyperino ψ_+ (ψ_-), and a complex auxiliary field \mathcal{F}^1 (\mathcal{F}^2)

$$\mathbf{H}^1 = H^1 + \sqrt{2}\theta\psi_+ + \theta^2\mathcal{F}^1, \quad \mathcal{F}^1 \equiv F^1 + (\mathcal{D}_y - \Sigma)H^2 + H^2 M, \quad (2.18)$$

¹We use the convention of Wess and Bagger [56] for Grassmann coordinates and superfields in this paper, except that four-dimensional spacetime indices are denoted by Greek alphabets $\mu, \nu = 0, \dots, 3$. For conventions of superfields in terms of component fields, we mostly follow those in Refs.[53], and [54].

$$\mathbf{H}^2 = H^2 + \sqrt{2}\bar{\theta}\bar{\psi}_- + \bar{\theta}^2\mathcal{F}^2, \quad \mathcal{F}^2 \equiv -F^2 - (\mathcal{D}_y + \Sigma)H^1 + H^1M, \quad (2.19)$$

where the auxiliary field \mathcal{F}^1 (\mathcal{F}^2) of the superfield for four SUSY is shifted from the auxiliary field F^1 (F^2) for eight SUSY by the covariant derivative of the other hypermultiplet scalar H^2 (H^1) and other² terms [53], [54]. Please note that we have chosen to denote the anti-chiral superfield as \mathbf{H}^2 , as shown in the $\bar{\theta}$ dependence in Eq.(2.19).

The complexified $U(N_C)$ gauge transformation is given in terms of the chiral scalar superfield Λ for the gauge parameter by [53], [54]

$$e^{2\mathbf{V}} \rightarrow e^{2\mathbf{V}'} = e^{\Lambda^\dagger} e^{2\mathbf{V}} e^\Lambda, \quad \Phi \rightarrow \Phi' = e^{-\Lambda}\Phi e^\Lambda + e^{-\Lambda}\partial_y(e^\Lambda), \quad (2.20)$$

$$\mathbf{H}^1 \rightarrow \mathbf{H}^{1'} = e^{-\Lambda}\mathbf{H}^1, \quad \mathbf{H}^2 \rightarrow \mathbf{H}^{2'} = e^{\Lambda^\dagger}\mathbf{H}^2. \quad (2.21)$$

Then their infinitesimal transformations $\delta_G^C(\Lambda)$ become

$$\delta_G^C(\Lambda)e^{2\mathbf{V}} = \Lambda^\dagger e^{2\mathbf{V}} + e^{2\mathbf{V}}\Lambda, \quad \delta_G^C(\Lambda)\Phi = -[\Lambda, \Phi] + \partial_y\Lambda, \quad (2.22)$$

$$\delta_G^C(\Lambda)\mathbf{H}^1 = -\Lambda\mathbf{H}^1, \quad \delta_G^C(\Lambda)\mathbf{H}^2 = \Lambda^\dagger\mathbf{H}^2. \quad (2.23)$$

Using the infinitesimal form of the gauge transformation, we can define the derivative \hat{D}_y which is covariant under the complexified gauge transformations

$$\hat{D}_y \equiv \partial_y - \delta_G^C(\Phi), \quad (2.24)$$

For example, the covariant derivative for the hypermultiplet \mathbf{H}^1 and the adjoint chiral scalar multiplet Φ are given by

$$\begin{aligned} \hat{D}_y\mathbf{H}^1 &= (\partial_y + \Phi)\mathbf{H}^1, \\ \hat{D}_ye^{2\mathbf{V}} &= \partial_ye^{2\mathbf{V}} - \Phi^\dagger e^{2\mathbf{V}} - e^{2\mathbf{V}}\Phi. \end{aligned} \quad (2.25)$$

If supplemented by fermionic terms, the bosonic Lagrangian (2.1) becomes invariant under the supersymmetric transformations with eight (real) Grassmann parameters. We can now rewrite this fundamental Lagrangian \mathcal{L} in terms of the superfields for four supercharges as

$$\begin{aligned} \mathcal{L} &= -\mathcal{E}_w + \int d^4\theta \text{Tr} \left[-2c\mathbf{V} + \frac{1}{2g^2} \left(e^{-2\mathbf{V}} \hat{D}_ye^{2\mathbf{V}} \right)^2 + e^{2\mathbf{V}}\mathbf{H}^1\mathbf{H}^{1\dagger} + e^{-2\mathbf{V}}\mathbf{H}^2\mathbf{H}^{2\dagger} \right] \\ &+ \left(\int d^2\theta \text{Tr} \left[\hat{D}_y\mathbf{H}^1\mathbf{H}^{2\dagger} - \mathbf{H}^1M\mathbf{H}^{2\dagger} + \frac{1}{4g^2}\mathbf{W}^\alpha\mathbf{W}_\alpha \right] + \text{h.c.} \right), \end{aligned} \quad (2.26)$$

where field strength superfield \mathbf{W} is given by

$$\mathbf{W}_\alpha \equiv -\frac{1}{8}\bar{D}\bar{D}e^{-2\mathbf{V}}D_\alpha e^{2\mathbf{V}}. \quad (2.27)$$

In transforming the fundamental Lagrangian (2.26) in terms of the superfield for four SUSY into the manifestly supersymmetric form for eight SUSY (2.1), we need to make several partial integrations with respect to the fifth coordinate y , and have to retain the surface terms carefully

²The other terms involving the adjoint scalar Σ and the hypermultiplet mass matrix M can be understood as a result of the Scherk-Schwarz dimensional reduction from six dimensions.

in the procedure. We also note that the auxiliary fields for four SUSY \mathcal{Y}^3 , and \mathcal{F}^i are different from those for eight SUSY Y^3 , F^i by total derivative terms as in Eqs.(2.16), (2.18), and (2.19). In this way we find a total divergence \mathcal{E}_w representing the topological charge contributing to the energy density of the background which maintains four SUSY. Since we are interested in bosonic components of the topological term \mathcal{E} , we exhibit only the bosonic terms explicitly

$$\begin{aligned} \mathcal{E}_w = \partial_y \Big[& \text{Tr} [c\Sigma - (\Sigma H^1 H^{1\dagger} - H^1 M H^{1\dagger}) + (\Sigma H^2 H^{2\dagger} - H^2 M H^{2\dagger}) \\ & - \frac{2}{g^2} \mathcal{Y}^3 \Sigma + \mathcal{F}^1 H^{2\dagger} + H^2 \mathcal{F}^{1\dagger} + (\text{fermionic terms})] \Big]. \end{aligned} \quad (2.28)$$

Let us emphasize again that the topological term is precisely the difference between the fundamental Lagrangian which is manifestly supersymmetric under the eight SUSY and another fundamental Lagrangian in terms of superfields for four manifest SUSY.

Since we are interested in co-dimension one solitons, we are primarily considering (multiple) domain walls as the 1/2 BPS background. The solution typically have a number of parameters such as the position of walls. We call these parameters as moduli which are denoted as ϕ^i . Following the usual procedure[46], we promote these moduli to fields $\phi^i(x)$ on the world volume of the background soliton, and assume that the moduli fields $\phi^i(x)$ around the wall background to fluctuate only very slowly. Namely, we introduce a parameter λ for the slow movement and neglect high energy fluctuations as explained in sect.2.1. By explicitly writing the derivatives of moduli fields we obtain

$$\partial_y \phi^i = \mathcal{O}(1) \phi^i, \quad \partial_\mu \phi^i = \mathcal{O}(\lambda) \phi^i, \quad \lambda \ll \min(\Delta m, g\sqrt{c}). \quad (2.29)$$

Here and in the following, $\mathcal{O}(1)$ means that it is of the order of the characteristic mass scale $\min(\Delta m, g\sqrt{c})$. The slow-movement parameter λ in Eq.(2.29) is defined to be of the order of the world-volume-coordinate derivative ∂_μ . The supertransformation implies that the square of the derivative in terms of the Grassmann coordinates θ gives translation in the world-volume : $(\partial/\partial\theta)^2 \sim \partial_\mu$. Therefore we obtain

$$d\theta \sim \frac{\partial}{\partial\theta} \sim \mathcal{O}(\lambda^{\frac{1}{2}}). \quad (2.30)$$

To assign the order of λ for hypermultiplets, we observe that the first hypermultiplet H^1 has nonvanishing values whereas the second hypermultiplet H^2 vanishes in the 1/2 BPS background solution (2.7). If we let the moduli parameters to fluctuate over the world-volume coordinates with the order of λ , the fluctuation induces terms of order λ in both hypermultiplets. Therefore the second hypermultiplet H^2 naturally becomes nonvanishing values and is of order λ . Combining the above order estimates of component fields, we assume that the hypermultiplet superfields are of order

$$\mathbf{H}^1 \sim \mathcal{O}(1), \quad \mathbf{H}^2 \sim \mathcal{O}(\lambda). \quad (2.31)$$

Note that this assignment breaks half of supersymmetry, and surviving supersymmetry is manifest in this superfield formalism. BPS equations for walls also respect this supersymmetry automatically, as we will explain later. Similarly, the adjoint scalar Σ has nonvanishing values in the 1/2 BPS background solution (2.6)

$$\Phi \sim \mathcal{O}(1). \quad (2.32)$$

On the other hand, the gauge field W_μ vanishes in the BPS background, and only induced by the order λ fluctuations of moduli fields. Since the gauge field appears as the coefficient of $\bar{\theta}\gamma^\mu\theta \sim \mathcal{O}(\lambda^{-1})$, we find the vector multiplet to be of the order of

$$\mathbf{V} \sim \mathcal{O}(1), \quad (W_\mu \sim \mathcal{O}(\lambda)). \quad (2.33)$$

Then the kinetic terms for \mathbf{V}, \mathbf{H}^2 become of the order of

$$\int d^2\theta \text{Tr} W^\alpha W_\alpha \sim \mathcal{O}(\lambda^4), \quad \int d^4\theta \text{Tr} e^{-2\mathbf{V}} \mathbf{H}^2 \mathbf{H}^{2\dagger} \sim \mathcal{O}(\lambda^4). \quad (2.34)$$

Since we are interested in obtaining the leading nontrivial terms in the expansion in powers of λ , we retain up to two powers of λ corresponding to the nonlinear kinetic terms for the moduli fields. Neglecting $\mathcal{O}(\lambda^4)$ we obtain

$$\begin{aligned} \mathcal{L} = & -\mathcal{E}_w + \int d^4\theta \text{Tr} \left[-2c\mathbf{V} + e^{2\mathbf{V}} \mathbf{H}^1 \mathbf{H}^{1\dagger} + \frac{1}{2g^2} \left(e^{-2\mathbf{V}} \hat{D}_y e^{2\mathbf{V}} \right)^2 \right] \\ & + \left(\int d^2\theta \text{Tr} \left[\hat{D}_y \mathbf{H}^1 \mathbf{H}^{2\dagger} - \mathbf{H}^1 M \mathbf{H}^{2\dagger} \right] + \text{h.c.} \right). \end{aligned} \quad (2.35)$$

Up to this order, we can see that \mathbf{H}^2, \mathbf{V} serve as Lagrange multiplier fields. Namely the field equations for \mathbf{H}^2 and \mathbf{V} give constraints

$$\hat{D}_y \mathbf{H}^1 = \mathbf{H}^1 M, \quad (2.36)$$

$$g^2(c - \mathbf{H}^1 \mathbf{H}^{1\dagger} e^{2\mathbf{V}}) = -\hat{D}_y \left(e^{-2\mathbf{V}} \hat{D}_y e^{2\mathbf{V}} \right), \quad (2.37)$$

respectively. These constraint equations are in terms of superfields. To clarify the physical content of the constraint equations (2.36) and (2.37), let us expand them in powers of the Grassmann coordinates $\theta, \bar{\theta}$. The first constraint equation (2.36) gives the BPS equation (2.3) for the hypermultiplet H^1 as the lowest term (independent of $\theta, \bar{\theta}$). For the second constraint equation (2.37), it is useful to multiply $e^{\mathbf{V}}$ and $e^{-\mathbf{V}}$ from left and right. Then the left-hand side becomes

$$e^{\mathbf{V}} g^2 (c - \mathbf{H}^1 \mathbf{H}^{1\dagger} e^{2\mathbf{V}}) e^{-\mathbf{V}} = g^2 \left((c - H^1 H^{1\dagger}) + \theta \sigma^\mu \bar{\theta} i (\mathcal{D}_\mu H^1 H^{1\dagger} - H^1 \mathcal{D}_\mu H^{1\dagger}) \right) + \dots, \quad (2.38)$$

where dots denote terms with other combinations of θ , and terms bilinear in fermions are neglected. The right-hand side of Eq.(2.37) becomes

$$-e^{\mathbf{V}} \hat{D}_y \left(e^{-2\mathbf{V}} \hat{D}_y e^{2\mathbf{V}} \right) e^{-\mathbf{V}} = 2 \left(\mathcal{D}_y \Sigma - \theta \sigma^\mu \bar{\theta} (\mathcal{D}_y F_{\mu y} + i[\Sigma, \mathcal{D}_\mu \Sigma]) \right) + \dots. \quad (2.39)$$

The lowest component gives the BPS equation (2.4) for vector multiplet scalar Σ with $H^2 = 0$. The BPS equation for the second hypermultiplet H^2 does not follow from the above constraint equation. This is natural since our choice of the preserved four supercharges implies $H^2 \equiv 0$ as the solution of BPS equations. By comparing the coefficients of $\theta \sigma^\mu \bar{\theta}$ in Eqs.(2.38) and (2.39), we find that the vector component of Eq.(2.37) precisely gives the field equation (2.15) for the gauge field W_μ which has to be imposed to obtain the configuration of the gauge field W_μ . Thus we see that all the necessary informations to determine the field configurations including fluctuations are contained systematically in this superfield formulation. It is worth emphasizing that even the

BPS equations follow without using the usual method like the Bogomol'nyi completion, and that it is just an automatic consequence of the λ expansion with four manifest SUSY.

So far we have not specified any gauge of the complexified $U(N_C)$ local gauge invariance in Eqs.(2.20) and (2.21). In the spirit of our method in solving BPS equations, we can first solve the constraint equation (2.36) for the hypermultiplet, and reformulate the rest into a gauge invariant fashion. Let us define an element of the complexified gauge transformation \mathbf{S} to express the chiral scalar superfield Φ for the adjoint scalar of the vector multiplet as a pure gauge

$$\Phi = \mathbf{S}^{-1} \partial_y \mathbf{S}. \quad (2.40)$$

Then the constraint equation (2.36) for the hypermultiplet chiral superfield becomes simpler

$$\partial_y(\mathbf{S}\mathbf{H}^1) = \mathbf{S}\mathbf{H}^1 M, \quad (2.41)$$

which is easily solved in terms of the moduli matrix chiral superfields \mathbf{H}_0 as

$$\mathbf{H}^1(x, \theta, \bar{\theta}, y) = \mathbf{S}^{-1}(x, \theta, \bar{\theta}, y) \mathbf{H}_0(x, \theta, \bar{\theta}) e^{My}. \quad (2.42)$$

This solution clearly shows that the chiral superfield \mathbf{S} transforms under the complexified $U(N_C)$ gauge transformations in Eqs.(2.20) and (2.21) as

$$\mathbf{S} \rightarrow \mathbf{S}' = \mathbf{S} e^\Lambda. \quad (2.43)$$

On the other hand, there is an ambiguity to separate the chiral superfield \mathbf{S}^{-1} from the moduli matrix chiral superfield \mathbf{H}_0 in Eq.(2.42). If two sets of chiral superfields $(\mathbf{S}, \mathbf{H}_0)$ and $(\mathbf{S}', \mathbf{H}_0')$ are related by an element of \mathcal{V} of $GL(N_C, \mathbf{C})$

$$\mathbf{S}' = \mathcal{V} \mathbf{S}, \quad \mathbf{H}_0' = \mathcal{V} \mathbf{H}_0, \quad (2.44)$$

they give identical result for physical quantities such as hypermultiplet chiral superfield \mathbf{H}^1 . Thus the V -equivalence transformations [14], [15] in Eq.(2.8) are supersymmetrized by the chiral superfield \mathcal{V} .

After solving the hypermultiplet constraint equation (2.36), we can now define a vector superfield Ω which is invariant under the complexified $U(N_C)$ gauge transformations

$$\Omega \equiv \mathbf{S} e^{-2\mathbf{V}} \mathbf{S}^\dagger. \quad (2.45)$$

The remaining constraint equation (2.37) can be rewritten in terms of the gauge invariant superfield Ω as

$$\partial_y (\Omega^{-1} \partial_y \Omega) = g^2 c (1 - \Omega^{-1} \Omega_0), \quad \Omega_0 \equiv c^{-1} \mathbf{H}_0 e^{2My} \mathbf{H}_0^\dagger, \quad (2.46)$$

which gives the master equation (2.9) as the lowest component. Therefore this is the superfield extension of the master equation.

By using the solution of the constraint equation (2.36) for the hypermultiplet superfield \mathbf{H}^1 , we can rewrite the fundamental Lagrangian in Eq.(2.35) (up to order $\mathcal{O}(\lambda^2)$) in terms of the gauge invariant superfield Ω as

$$\mathcal{L} = -\mathcal{E}_w + \int d^4\theta \left[c \log \det \Omega + c \text{Tr} (\Omega_0 \Omega^{-1}) + \frac{1}{2g^2} \text{Tr} (\Omega^{-1} \partial_y \Omega)^2 \right] + \mathcal{O}(\lambda^4). \quad (2.47)$$

The first, second, and third terms in the $d^4\theta$ integrand come from the corresponding terms in the $d^4\theta$ integrand of the fundamental Lagrangian (2.35) (up to order $\mathcal{O}(\lambda^2)$). If we apply variational principle to find the minimum of the above five-dimensional Lagrangian (2.47) written in terms of Ω , we obtain the master equation (2.46) in terms of superfield. Since the fundamental Lagrangian (2.35) is the result of solving the hypermultiplet constraint (2.36), it is natural to expect that the master equation (2.46) is obtained by the action principle applied to the Lagrangian (2.35) in terms of Ω as the dynamical variable.

The superspace extension (2.46) of the master equation provides a method to determine all the quantities of interest as a systematic expansion in powers of Grassmann coordinates $\theta, \bar{\theta}$ as follows. Suppose we have an exact solution $\Omega_{\text{sol}}(H_0, H_0^\dagger, y)$ for the master equation (2.9) as a function of moduli matrix H_0, H_0^\dagger

$$\Omega = \Omega \Big|_{\theta=0} = \Omega_{\text{sol}}(H_0(x), H_0^\dagger(x), y). \quad (2.48)$$

By promoting the moduli matrix to a superfield $\mathbf{H}_0, \mathbf{H}_0^\dagger$ defined in Eq.(2.42), we obtain the solution for the vector superfield Ω of the superfield master equation (2.46) as a composite of the chiral and the anti-chiral superfields,

$$\Omega_{\text{sol}}(\mathbf{H}_0(x, \theta), \mathbf{H}_0^\dagger(x, \bar{\theta}), y) \equiv \Omega_{\text{sol}}. \quad (2.49)$$

As we noted in Eq.(2.45), the superfield $\Omega = \mathbf{S}e^{-2\mathbf{V}}\mathbf{S}^\dagger$ is $U(N_C)$ supergauge invariant, but the division between $\mathbf{S}, (\mathbf{S}^\dagger)$ and $e^{-2\mathbf{V}}$ depends on the gauge choice. In obtaining the solution for the fluctuation fields such as W_μ , we need to choose to use the Wess-Zumino gauge for the real general (vector) superfield \mathbf{V}_{sol} . This gauge transformation to the Wess-Zumino gauge is expressed as a multiplication of the chiral \mathbf{S}_{sol} and anti-chiral $\mathbf{S}_{\text{sol}}^\dagger$ superfields from left and right respectively as

$$\mathbf{S}_{\text{sol}}e^{-2\mathbf{V}_{\text{sol}}}\mathbf{S}_{\text{sol}}^\dagger = \Omega_{\text{sol}}. \quad (2.50)$$

Then expansion of the left-hand side of Eq.(2.50) in powers of the Grassmann coordinates $\theta, \bar{\theta}$ gives

$$\mathbf{S}_{\text{sol}}e^{-2\mathbf{V}_{\text{sol}}}\mathbf{S}_{\text{sol}}^\dagger = S_{\text{sol}}S_{\text{sol}}^\dagger + \theta\sigma^\mu\bar{\theta} \left(i(\partial_\mu S_{\text{sol}})S_{\text{sol}}^\dagger - iS_{\text{sol}}(\partial_\mu S_{\text{sol}}^\dagger) + 2S_{\text{sol}}W_\mu^{\text{sol}}S_{\text{sol}}^\dagger \right) + \cdots, \quad (2.51)$$

where we have not displayed the bilinear terms of fermions, and dots denote other powers of Grassmann coordinates. Expanding the right-hand side of Eq.(2.50) we obtain

$$\Omega_{\text{sol}}(\mathbf{H}_0, \mathbf{H}_0^\dagger, y) = \Omega_{\text{sol}} + \theta\sigma^\mu\bar{\theta} (i(\delta_\mu - \delta_\mu^\dagger)\Omega_{\text{sol}}) + \cdots, \quad (2.52)$$

we define the variation δ_μ and its conjugate δ_μ^\dagger with respect to the scalar fields of chiral superfields and anti-chiral superfields

$$\delta_\mu \equiv \sum_i \partial_\mu \phi^i \frac{\delta}{\delta \phi^i}, \quad \delta_\mu^\dagger \equiv \sum_i \partial_\mu \phi^{i*} \frac{\delta}{\delta \phi^{i*}}, \quad (2.53)$$

respectively. If the variation δ_μ and δ_μ^\dagger act on those functions which depend on the world-volume coordinates x^μ only through moduli fields, they satisfy

$$\partial_\mu = \delta_\mu + \delta_\mu^\dagger. \quad (2.54)$$

Comparing the lowest component of (2.51) and (2.52), we obtain

$$S_{\text{sol}} S_{\text{sol}}^\dagger = \Omega_{\text{sol}}. \quad (2.55)$$

This shows that we cannot avoid S_{sol} to depend on both ϕ^i and ϕ^{i*} , since we cannot factorize these dependences in Ω_{sol} . One should note that \mathbf{S}_{sol} ($\mathbf{S}_{\text{sol}}^\dagger$) is still chiral (anti-chiral) scalar superfield, taking both ϕ^i and ϕ^{i*} as lowest components of chiral scalar superfields. Comparison of the vector component of (2.51) and (2.52), we obtain a solution of the gauge fields as

$$-iW_\mu^{\text{sol}} = S_{\text{sol}}^{-1} \delta_\mu^\dagger S_{\text{sol}} + S_{\text{sol}}^\dagger \delta_\mu S_{\text{sol}}^{\dagger-1} + (\text{bi-linear terms of fermions}). \quad (2.56)$$

It is interesting to observe that this solution of gauge field fluctuation W_μ^{sol} receives contributions only from the ϕ^{i*} (ϕ^i) dependence of S_{sol} (S_{sol}^\dagger), in spite of the \mathbf{S}_{sol} being the chiral superfield. Similarly the adjoint scalar Σ and the gauge field W_y in the extra fifth direction is obtained from the lowest component of Eq.(2.40)

$$\Phi_{\text{sol}} = \mathbf{S}_{\text{sol}}^{-1} \partial_y \mathbf{S}_{\text{sol}} \quad \rightarrow \quad \Sigma^{\text{sol}} + iW_y^{\text{sol}} = S_{\text{sol}}^{-1} \partial_y S_{\text{sol}}. \quad (2.57)$$

The other components of the superfields Ω , \mathbf{V} , and Φ are similarly determined by the superfield equations.

In order to obtain the low-energy effective Lagrangian \mathcal{L}_{eff} finally, we just need to substitute the solutions Ω_{sol} into the fundamental Lagrangian \mathcal{L} and integrate over the extra dimensional coordinate y . The resulting four-dimensional effective Lagrangian for the moduli matrix superfield \mathbf{H}_0 is given by

$$\mathcal{L}_{\text{eff}} = \int dy \mathcal{L} = -T_w + \int d^4\theta K(\phi, \phi^*) + \mathcal{O}(\lambda^4), \quad (2.58)$$

where the Kähler potential is expressed by an integral form as

$$K(\phi, \phi^*) = \int dy \mathcal{K}(\phi, \phi^*, \Omega, y) \Big|_{\Omega=\Omega_{\text{sol}}}, \quad (2.59)$$

with a density

$$\mathcal{K}(\phi, \phi^*, \Omega, y) = -c \log \det \Omega + c \text{Tr} (\Omega_0 \Omega^{-1}) + \frac{1}{2g^2} \text{Tr} (\Omega^{-1} \partial_y \Omega)^2. \quad (2.60)$$

Since we are considering the massless fields corresponding to the moduli, we naturally obtain a nonlinear sigma model whose kinetic terms are specified by the Kähler potential $K(\phi, \phi^*)$, without any potential terms. Let us note that our method gives the density of the Kähler potential directly without going through the Kähler metric. This is in contrast to the component approach where one usually obtains the Kähler metric of the nonlinear sigma model with component scalar fields, such as in Eq.(A.8), and then integrate it to obtain the Kähler potential with a lot of labor.

When we consider a composite state of domain walls, vortices (or lumps) and monopoles [37], vortices and monopoles can be interpreted in the effective Lagrangian on the domain wall as follows: the first contribution corresponds to the energy of vortices (or lumps), and the third contribution corresponds to the energy of monopoles. It is interesting to note that our effective Lagrangian is not just an effective Lagrangian on a single wall, but an effective Lagrangian on the multiple wall system with various moduli such as relative distance moduli as the effective fields.

Therefore we can discuss lumps stretched between multiple walls, which was difficult previously [57].

By using the superfield master equation (2.46), we can show that the second term in Eq.(2.60) becomes a total derivative term. Therefore it can be omitted from the effective Lagrangian. The wall tension T_w is given by the topological charge as an integral over the total derivative term \mathcal{E}_w in Eq.(2.28)

$$T_w = \int dy \mathcal{E}_w = \text{Tr} [c\Sigma]_{y=-\infty}^{y=\infty}, \quad (2.61)$$

where we have used the boundary condition which requires that vacua are reached at both infinities $y = \pm\infty$.

If we take the strong coupling limit $g^2 \rightarrow \infty$, we find that the superfield master equation (2.46) becomes just an algebraic equation $\Omega = \Omega_0$. Therefore exact solutions for Ω can be obtained and the Kähler potential assumes a simple form in this case [14]

$$K_0(\phi, \phi^*) = c \int dy \log \det \Omega_0. \quad (2.62)$$

3 Superfield Effective Lagrangian on Vortices

As another example to illustrate the power of our method, we shall consider vortex as another 1/2 BPS soliton in this section. If we wish to obtain four-dimensional world-volume on vortices, we need to use a fundamental theory in six dimensions. Since this is the theory with eight SUSY in highest dimensions, all the lower-dimensional theories can be obtained by a simple dimensional reduction and/or a Scherk-Schwarz dimensional reduction with twisted boundary conditions along the compactified directions. We take again the $U(N_C)$ supersymmetric gauge theory with N_F (massless) hypermultiplets in the fundamental representation.³ Vortex solutions in six dimensions with manifest four supercharges in four-dimensional world-volume were discussed [31]. The bosonic part of the fundamental Lagrangian in six dimensions is given by

$$\begin{aligned} \mathcal{L}|_{\text{boson}} = & \text{Tr} \left[-\frac{1}{2g^2} F_{MN}(W) F^{MN}(W) + \frac{1}{g^2} \sum_{a=1}^3 (Y^a)^2 - cY^3 \right. \\ & \left. - \mathcal{D}^M H^i (\mathcal{D}_M H^i)^\dagger + F^i F^{i\dagger} + Y^a (\sigma^a)_{ij} H^j H^{i\dagger} \right], \end{aligned} \quad (3.1)$$

where the indices $M, N = 0, 1, \dots, 5$ run over six spacetime dimensions. The Lagrangian in five dimensions with massive hypermultiplets in Eq.(2.1) can be obtained by a Scherk-Schwarz dimensional reduction along the sixth direction x^5 .

The fundamental Lagrangian of the supersymmetric gauge theories in six dimensions can be written in terms of the superfields with four SUSY manifest [51]. We can use the same superfields as in five dimensions in Eqs.(2.16), (2.17), (2.18), (2.19) except for two features. The component fields should now depend on six dimensions rather than five dimensions, and the auxiliary fields for eight SUSY Y^3 and F^i are slightly more shifted from those for four supercharges \mathcal{Y}^3 and \mathcal{F}^i ,

³Hypermultiplets in six-dimensions must be massless.

compared to the five-dimensional case in Eqs.(2.16), (2.18) and (2.19). The chiral and anti-chiral superfields \mathbf{H}^1 and \mathbf{H}^2 for hypermultiplets are given by

$$\mathbf{H}^1 = H^1 + \sqrt{2}\theta\psi_+ + \theta^2\mathcal{F}^1, \quad \mathcal{F}^1 \equiv F^1 + (\mathcal{D}_4 + i\mathcal{D}_5)H^2, \quad (3.2)$$

$$\mathbf{H}^2 = H^2 + \sqrt{2}\bar{\theta}\bar{\psi}_- + \bar{\theta}^2\mathcal{F}^2, \quad \mathcal{F}^2 \equiv -F^2 - (\mathcal{D}_4 - i\mathcal{D}_5)H^1. \quad (3.3)$$

The chiral scalar superfield Φ containing the adjoint scalar is given by

$$\Phi = W_5 + iW_4 + \sqrt{2}\theta(-i\sqrt{2}\lambda_-) + \theta^2(Y^1 + iY^2), \quad (3.4)$$

where we identify the adjoint scalar field Σ to be the sixth component of the gauge field W_5 and denote the fifth component of gauge field as W_4 rather than W_y . Finally the vector superfield \mathbf{V} contains gauge field, gaugino, and auxiliary field and has a decomposition in the Wess-Zumino gauge

$$\mathbf{V}\Big|_{\text{WZ}} = -\theta\sigma^\mu\bar{\theta}W_\mu + i\theta^2\bar{\theta}\bar{\lambda}_+ - i\bar{\theta}^2\theta\lambda_+ + \frac{1}{2}\theta^2\bar{\theta}^2\mathcal{Y}^3, \quad \mathcal{Y}^3 \equiv Y^3 - F_{45}(W). \quad (3.5)$$

Let us define complex variables for extra two dimensions

$$z \equiv x^4 + ix^5, \quad \partial \equiv \frac{1}{2}(\partial_4 - i\partial_5). \quad (3.6)$$

The six-dimensional Lagrangian can be rewritten in terms of these superfields for four SUSY. It contains a term \mathcal{L}_{sym} similar to that in the five-dimensional case

$$\begin{aligned} \mathcal{L}_{\text{sym}} = \int d^4\theta \text{Tr} & \left[-2c\mathbf{V} + \frac{1}{g^2} (2e^{-2\mathbf{V}}\bar{\partial}e^{2\mathbf{V}}e^{-2\mathbf{V}}\partial e^{2\mathbf{V}} - 2\partial e^{2\mathbf{V}}e^{-2\mathbf{V}}\Phi^\dagger \right. \\ & \left. - 2e^{-2\mathbf{V}}\bar{\partial}e^{2\mathbf{V}}\Phi + e^{-2\mathbf{V}}\Phi^\dagger e^{2\mathbf{V}}\Phi) + e^{2\mathbf{V}}\mathbf{H}^1\mathbf{H}^{1\dagger} + e^{-2\mathbf{V}}\mathbf{H}^2\mathbf{H}^{2\dagger} \right] \\ & + \left(\int d^2\theta \text{Tr} \left[(2\partial + \Phi)\mathbf{H}^1\mathbf{H}^{2\dagger} + \frac{1}{4g^2}W^\alpha W_\alpha \right] + \text{h.c.} \right). \end{aligned} \quad (3.7)$$

However, it has been known that the fundamental Lagrangian with eight SUSY in six-dimensions requires an addition of a Wess-Zumino-Witten like term \mathcal{L}_{WZW} in order to maintain the complexified $U(N_C)$ gauge invariance if superfields for four SUSY are used [51], [48]

$$\mathcal{L}_{\text{WZW}} \equiv \int d^4\theta \text{Tr} \left[\frac{16}{g^2} \bar{\partial}\mathbf{V} \frac{\sinh(2L_{\mathbf{V}}) - 2L_{\mathbf{V}}}{(2L_{\mathbf{V}})^2} \partial\mathbf{V} \right], \quad (3.8)$$

where the operation $L_{\mathbf{V}}$ is defined by

$$L_{\mathbf{V}} \times \mathbf{X} = [\mathbf{V}, \mathbf{X}]. \quad (3.9)$$

To rewrite the fundamental six-dimensional Lagrangian \mathcal{L} in Eq.(3.1) in terms of the superfields for four SUSY, we need to retain total divergence terms $-\mathcal{E}_v$ when we make partial integrations in the codimensions of the soliton, namely fifth and sixth directions, similarly to the wall case in five dimensions

$$\mathcal{L} = -\mathcal{E}_v + \mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{WZW}}. \quad (3.10)$$

It is easiest to evaluate the total divergence term \mathcal{E}_v in the Wess-Zumino gauge (3.5) where the Wess-Zumino-Witten like term in Eq.(3.8) vanishes $\mathcal{L}_{\text{WZW}} = 0$. We also need to take account of the difference of auxiliary fields for four supercharges \mathcal{Y}^3 , and \mathcal{F}^i from those for eight SUSY Y^3 , F^i in Eqs.(3.5), (3.2), and (3.3). We obtain the total divergence term \mathcal{E}_v representing the topological charge contributing to the energy density of the (vortex) background which maintains four SUSY, exhibiting only the bosonic terms explicitly as

$$\begin{aligned} \mathcal{E}_v = & \text{Tr} \left[cF_{45}(W) - \frac{2}{g^2} (\partial_4(\mathcal{Y}^3 W_5) - \partial_5(\mathcal{Y}^3 W_4)) + 2\partial(\mathcal{F}^1 H^{2\dagger}) + 2\bar{\partial}(H^2 \mathcal{F}^{1\dagger}) \right. \\ & \left. + i(\partial_4(H^{1\dagger} \mathcal{D}_5 H^1 - H^{2\dagger} \mathcal{D}_5 H^2) - \partial_5(H^{1\dagger} \mathcal{D}_4 H^1 - H^{2\dagger} \mathcal{D}_4 H^2)) \right]. \end{aligned} \quad (3.11)$$

Let us observe that only the first term contributes to the energy when the total divergence term is integrated over the two-dimensional plane, since we require that the field configuration should reduce to vacuum at spacial infinity where four SUSY are maintained: $\mathcal{Y}^3 = 0$, and $\mathcal{F}^i = 0$.

Although the Wess-Zumino gauge is useful to reveal the physical field content, it is more useful to consider generic gauges in order to perform a systematic expansion in powers of the slow-movement parameter λ . If we do not choose the Wess-Zumino gauge, we need to retain the Wess-Zumino-Witten like term. As shown in Appendix B, the first term $(2e^{-2\mathbf{V}} \bar{\partial} e^{2\mathbf{V}} e^{-2\mathbf{V}} \partial e^{2\mathbf{V}})$ in the bracket (multiplied by $1/g^2$) of Eq.(3.7) can be combined with the Wess-Zumino-Witten-like term (3.8) to give the following total fundamental Lagrangian as

$$\begin{aligned} \mathcal{L} = & -\mathcal{E}_v + \int d^4\theta \text{Tr} \left[-2c\mathbf{V} + \frac{1}{g^2} \left(16 \int_0^1 dx \int_0^x dy \bar{\partial} \mathbf{V} e^{2yL\mathbf{V}} \partial \mathbf{V} - 2\partial e^{2\mathbf{V}} e^{-2\mathbf{V}} \Phi^\dagger \right. \right. \\ & \left. \left. - 2e^{-2\mathbf{V}} \bar{\partial} e^{2\mathbf{V}} \Phi + e^{-2\mathbf{V}} \Phi^\dagger e^{2\mathbf{V}} \Phi \right) + e^{2\mathbf{V}} \mathbf{H}^1 \mathbf{H}^{1\dagger} + e^{-2\mathbf{V}} \mathbf{H}^2 \mathbf{H}^{2\dagger} \right] \\ & + \left(\int d^2\theta \text{Tr} \left[(2\partial + \Phi) \mathbf{H}^1 \mathbf{H}^{2\dagger} + \frac{1}{4g^2} W^\alpha W_\alpha \right] + \text{h.c.} \right). \end{aligned} \quad (3.12)$$

Let us find out the order of λ for various fields. It is well-known that the first hypermultiplet H^1 has non-vanishing values, whereas the second hypermultiplet H^2 vanishes [29]–[32]. When we promote the moduli parameters to fields on the world volume of the soliton, the fluctuation naturally induces terms of order λ to both H^1, H^2 . Therefore we need to assume the same order assignment for various fields as in the wall case (2.31), (2.32), and (2.33). Now let us expand the fundamental Lagrangian in powers of the slow-movement parameter λ . If we retain terms up to the order of $\mathcal{O}(\lambda^2)$, we obtain

$$\begin{aligned} \mathcal{L} = & -\mathcal{E}_v + \int d^4\theta \text{Tr} \left[-2c\mathbf{V} + \frac{1}{g^2} \left(16 \int_0^1 dx \int_0^x dy \bar{\partial} \mathbf{V} e^{2yL\mathbf{V}} \partial \mathbf{V} - 2\partial e^{2\mathbf{V}} e^{-2\mathbf{V}} \Phi^\dagger \right. \right. \\ & \left. \left. - 2e^{-2\mathbf{V}} \bar{\partial} e^{2\mathbf{V}} \Phi + e^{-2\mathbf{V}} \Phi^\dagger e^{2\mathbf{V}} \Phi \right) + e^{2\mathbf{V}} \mathbf{H}^1 \mathbf{H}^{1\dagger} \right] \\ & + \left(\int d^2\theta \text{Tr} \left[(2\partial + \Phi) \mathbf{H}^1 \mathbf{H}^{2\dagger} \right] + \text{h.c.} \right) + \mathcal{O}(\lambda^4). \end{aligned} \quad (3.13)$$

The superfield \mathbf{H}^2 now acts as a Lagrange multiplier. The constraint resulting from varying \mathbf{H}^2 gives the BPS equation for hypermultiplet \mathbf{H}^1 in the superfield form

$$(2\partial + \Phi) \mathbf{H}^1 = 0. \quad (3.14)$$

This BPS equation can be easily solved in terms of a complexified gauge transformation superfield \mathbf{S} defined as

$$\Phi = 2\mathbf{S}^{-1}\partial\mathbf{S}, \quad \mathbf{H}^1(x^\alpha, \theta, \bar{\theta}, z, \bar{z}) = \mathbf{S}^{-1}(x^\alpha, \theta, \bar{\theta}, z, \bar{z})\mathbf{H}_0(x^\alpha, \theta, \bar{\theta}, \bar{z}). \quad (3.15)$$

Similarly to the wall case, we can define a gauge invariant vector superfield Ω as

$$\Omega(x^\alpha, \theta, \bar{\theta}, z, \bar{z}) \equiv \mathbf{S}(x^\alpha, \theta, \bar{\theta}, z, \bar{z})e^{-2\mathbf{V}}\mathbf{S}^\dagger(x^\alpha, \theta, \bar{\theta}, z, \bar{z}). \quad (3.16)$$

Another Lagrange multiplier superfield \mathbf{V} in Eq.(3.13) gives another constraint

$$\begin{aligned} g^2(c - \mathbf{H}^1\mathbf{H}^{1\dagger}e^{2\mathbf{V}}) &= -2\partial(e^{-2\mathbf{V}}2\bar{\partial}e^{2\mathbf{V}} - 2\Phi^\dagger e^{2\mathbf{V}}) - 2\bar{\partial}\Phi \\ &+ \Phi(e^{-2\mathbf{V}}2\bar{\partial}e^{2\mathbf{V}} - 2\Phi^\dagger e^{2\mathbf{V}}) - (e^{-2\mathbf{V}}2\bar{\partial}e^{2\mathbf{V}} - 2\Phi^\dagger e^{2\mathbf{V}})\Phi. \end{aligned} \quad (3.17)$$

This superfield constraint contains the BPS equation for vector multiplet as the lowest component (independent of the Grassmann coordinate θ)

$$g^2(c - H^1 H^{1\dagger}) = 2F_{45}(W). \quad (3.18)$$

We can rewrite the constraint equation in terms of the gauge invariant superfield Ω as

$$g^2c(1 - \Omega_0\Omega^{-1}) = 2\partial(2\bar{\partial}\Omega\Omega^{-1}), \quad (3.19)$$

whose lowest component (independent of the Grassmann coordinate θ) is precisely the master equation in the case of vortices [34].⁴ Changing variables from Φ to \mathbf{S} is nothing but a complexified $U(N_C)$ gauge transformation to choose a gauge where Φ is eliminated. In that gauge, we obtain $e^{-2\mathbf{V}} \rightarrow \Omega$, $\mathbf{H}^1 \rightarrow \mathbf{H}_0$.

In order to obtain the effective Lagrangian up to two derivatives, we solve the constraint equation (3.14) for hypermultiplet superfield \mathbf{H}^1 and use the solution to rewrite the fundamental Lagrangian in Eq.(3.13) (up to order $\mathcal{O}(\lambda^2)$) in terms of the gauge invariant superfield Ω . After integrating over two extra dimensions x^4, x^5 , we obtain the effective Lagrangian for moduli fields whose nonlinear kinetic term is described in terms of the Kähler potential K apart from the energy of the background 1/2 BPS vortices T_v

$$\mathcal{L}_{\text{eff}} = \int dx^4 dx^5 \mathcal{L} = -T_v + \int d\theta^4 K(\phi, \phi^*) + \mathcal{O}(\lambda^4), \quad (3.20)$$

$$T_v = \int dx^4 dx^5 \mathcal{E}_v = \int dx^4 dx^5 \text{Tr}[cF_{45}(W)]. \quad (3.21)$$

To obtain the Kähler potential we should evaluate the following general formula

$$K = \int dx^4 dx^5 \text{Tr} \left[-2c\mathbf{V} + e^{2\mathbf{V}}\mathbf{H}_0\mathbf{H}_0^\dagger + \frac{16}{g^2} \int_0^1 dx \int_0^x dy \bar{\partial}\mathbf{V} e^{2yL_v} \partial\mathbf{V} \right], \quad (3.22)$$

where the integration over the x^4 - x^5 plane may require regularization. However, we believe that the divergent pieces can only come from two possible sources: the first possibility is the terms

⁴The master equation reduces to the so-called Taubes equation [26] in the case of Abrikosov-Nielsen-Olesen vortices [24, 25] ($N_C = N_F = 1$) by rewriting $c\Omega(z, \bar{z}) = |H_0|^2 e^{-\xi(z, \bar{z})}$ with $H_0 = \prod_i (z - z_i)$. Note that $\log \Omega$ is regular everywhere while ξ is singular at vortex positions.

that can be gauged away as Kähler transformations which does not affect the physical Kähler metric, and the second possibility is the non-normalizable modes which have support extending to infinity and should be excluded from the dynamical variable in the effective Lagrangian after all. The Kähler metric [27] and its potential [28] for the Abelian gauge theory has been obtained before. Let us again emphasize that the above formula for the Kähler potential density is obtained without the need of computing the Kähler metric first in contrast to the component approach.

In order to obtain the Kähler metric explicitly, we just need to vary this Kähler potential. In varying the Kähler potential, one should be careful to interchange the variation and the integration over the x^4 - x^5 plane⁵, because of the possible necessity of regularization [28]. Assuming the variation can be interchanged with the integration, the variation of this Kähler potential of the effective Lagrangian can be expressed in terms of the gauge invariant superfield $\Omega = e^{-2V}$ as

$$\delta K = \int dx^4 dx^5 \text{Tr} \left[\Omega^{-1} \delta \Omega \left\{ c(\mathbf{1}_{N_C} - \Omega_0 \Omega^{-1}) - \frac{4}{g^2} \partial (\bar{\partial} \Omega \Omega^{-1}) \right\} + c \Omega^{-1} \delta \Omega_0 \right]. \quad (3.23)$$

We need to express all the superfields in terms of the moduli fields by substituting them with the solution of the constraint equation (3.19), namely the master equation. Then the terms in the curly bracket vanish and the Ω_0 in the last term can also be rewritten in terms of the solution Ω as

$$\begin{aligned} \delta K \Big|_{\Omega=\Omega_{\text{sol}}} &= \int dx^4 dx^5 \text{Tr} [c \Omega^{-1} \delta \Omega_0] \Big|_{\Omega=\Omega_{\text{sol}}} \\ &= \int dx^4 dx^5 \text{Tr} \left[-\frac{4}{g^2} \delta \left\{ \partial (\bar{\partial} \Omega \Omega^{-1}) \right\} + \delta \Omega \Omega^{-1} \left\{ c - \frac{4}{g^2} \partial (\bar{\partial} \Omega \Omega^{-1}) \right\} \right] \Big|_{\Omega=\Omega_{\text{sol}}}. \end{aligned} \quad (3.24)$$

By choosing the variation δ as the variation with respect to (the scalar fields in) chiral superfields δ^μ defined in Eq.(2.53), and by varying once more by its conjugate δ_μ^\dagger with respect to anti-chiral superfields, we can express the resulting Kähler metric more explicitly in terms of the gauge invariant quantity Ω appearing in the master equation (3.19)

$$\begin{aligned} \delta_\mu^\dagger \delta_\mu K \Big|_{\Omega=\Omega_{\text{sol}}} &= \int dx^4 dx^5 \text{Tr} [\delta_\mu^\dagger \delta_\mu c \log \Omega \\ &\quad + \frac{4}{g^2} \{ \partial (\delta^\mu \Omega \Omega^{-1}) \delta_\mu^\dagger (\bar{\partial} \Omega \Omega^{-1}) - \partial (\bar{\partial} \Omega \Omega^{-1}) \delta_\mu^\dagger (\delta^\mu \Omega \Omega^{-1}) \}] \Big|_{\Omega=\Omega_{\text{sol}}}, \end{aligned} \quad (3.25)$$

where we used an identity

$$\begin{aligned} &\delta^\dagger [\Omega \delta \Omega^{-1} \partial (\bar{\partial} \Omega \Omega^{-1})] + \partial [\delta \Omega \Omega^{-1} \delta^\dagger (\bar{\partial} \Omega \Omega^{-1})] \\ &= \partial (\delta \Omega \Omega^{-1}) \delta^\dagger (\bar{\partial} \Omega \Omega^{-1}) - \partial (\bar{\partial} \Omega \Omega^{-1}) \delta^\dagger (\delta \Omega \Omega^{-1}), \end{aligned} \quad (3.26)$$

and omitted a surface term $\partial [\delta \Omega \Omega^{-1} \delta^\dagger (\bar{\partial} \Omega \Omega^{-1})]$ by using $\delta \Omega \Omega^{-1} = 0$ at $x^4, x^5 \rightarrow \infty$. Except for the conventions differences, this Kähler metric for the vortex effective theory agrees with our previous result in Ref.[59].

⁵We thank Nick Manton for explaining this point to us.

4 Effective Lagrangian of Multi-wall System

In this section, we will perform the integral of the formulas (2.59) and (2.60) of Kähler potential for some examples of multi-wall systems to obtain explicit effective Lagrangians. In this stage, we should remove fake divergences contained in the formula by use of the Kähler transformation. Physical divergences implying non-normalizable modes should emerge if there exist moduli in vacua of the original Yang-Mills-Higgs system, whereas we do not consider such cases here.

4.1 Coordinates for the Moduli Space

We apply our method to the Abelian gauge theories with N_F flavors. In order to be facilitate an explicit evaluation of the integral, let us consider the case where the mass differences of hypermultiplets are quantized as

$$M = \text{diag}(m_1, \dots, m_{N_F}) \simeq m \text{diag}(n_1, n_2, \dots, n_{N_F-1}, 0), \quad n_A \in \mathbf{Z}_+, \quad (4.1)$$

with $n_A > n_{A+1}$, ($n_{N_F} = 0$). By using the V -equivalence transformations in Eq.(2.8) [14], we can choose a parametrization of the moduli matrix H_0 in terms of $N_F - 1$ complex moduli parameters τ^A as

$$H_0 = \sqrt{c} (1, \tau^2, \tau^3, \dots, \tau^{N_F}), \quad (\tau^1 = 1) \quad (4.2)$$

where $\tau^A \in \mathbf{C}$ for $2 \leq A \leq N_F - 1$ and $\tau^{N_F} \in \mathbf{C} - \{0\}$. The complex moduli parameters are the coordinates for the moduli space of multi-wall configurations. With this moduli matrix parametrization, the source term Ω_0 of the master equation (2.9) is given by

$$\Omega_0(y) = \sum_{A=1}^{N_F} |\tau^A|^2 w^{n_A} \equiv P(w), \quad w \equiv e^{2my}. \quad (4.3)$$

with a polynomial $P(w)$ of order n_1 . Vacua can be characterized by the flavors of the nonvanishing hypermultiplets. The N_F terms $|\tau^A|^2 w^{n_A}$ in Ω_0 represent weights of N_F vacua, which change as y varies. The wall is located at the position where weights of two adjacent vacua become equal. Thus, the position y_A of the wall interpolating the A -th vacuum and the $(A+1)$ -th vacuum is estimated as

$$y_A \approx -\frac{1}{m} \text{Re} \frac{\log \tau^A - \log \tau^{A+1}}{n_A - n_{A+1}} \quad (4.4)$$

The moduli parameters $\{\tau^A\}$ have a physical meaning of the positions of the wall and the phase of the vacua.

4.2 Integral at the Strong Gauge Coupling Limit

By taking the strong gauge coupling limit $g^2 \rightarrow \infty$, the master equation (2.9) can be solved algebraically to give the solution

$$\Omega(y) = \Omega_0(y) = P(e^{2my}). \quad (4.5)$$

As a function of a complex variable w , the polynomial $P(w)$ has n_1 possible complex zeros $P(w_k) = 0$, and thus

$$P(w) = \prod_{k=1}^{n_1} (w - w_k). \quad (4.6)$$

By substituting this solution to the formula (2.62) and taking an infrared cutoff $2mL$, the Kähler potential is rewritten as

$$K_0 = \lim_{L \rightarrow \infty} \frac{c}{2m} \sum_{k=1}^{n_1} \int_{-L}^L dt \log(e^t - w_k). \quad (4.7)$$

The integral in the right hand side can be performed with the help of the following identity

$$\int_{-L}^L dt \log(e^t + e^s) = \frac{1}{2}(L + s)^2 + \frac{\pi}{6} + \mathcal{O}(e^{-L}), \quad (4.8)$$

to result in

$$K_0 = \lim_{L \rightarrow \infty} \frac{c}{2m} \left\{ \sum_{k=1}^{n_1} \frac{1}{2} [\log(-w_k)]^2 + L \sum_{k=1}^{n_1} \log(-w_k) + n_1 \left(\frac{L^2}{2} + \frac{\pi}{6} \right) + \mathcal{O}(e^{-L}) \right\}. \quad (4.9)$$

Comparing Eqs.(4.3) and (4.6) we obtain $\prod_{k=1}^{n_1} (-w_k) = |\tau^{N_F}|^2$, that is,

$$\sum_{k=1}^{n_1} \log(-w_k) = \log \tau^{N_F} + \log(\tau^{N_F})^*. \quad (4.10)$$

Therefore, the second and the third terms in right hand side of Eq.(4.9) can be eliminated by the Kähler transformation, and we obtain a convergent simple result

$$K_0(\tau^A, \tau^{A\dagger}) = \frac{c}{4m} \sum_{k=1}^{n_1} [\log(-w_k)]^2. \quad (4.11)$$

Here note that the quantities w_k defined by (4.6) are highly non-trivial functions of $|\tau^A|^2$. For instance, let us take the case of $N_F = 3$ with $(n_1, n_2, n_3) = (2, 1, 0)$, and change the parametrization of the moduli matrix H_0 from τ^2, τ^3 to two complex moduli parameters ϕ_+, ϕ_- as

$$H_0 = \sqrt{c} (1, \tau^2, \tau^3) = \sqrt{c} \left(1, e^{\frac{\phi_+ + \phi_-}{2}}, e^{\phi_+} \right). \quad (4.12)$$

We find that the formula (4.11) leads straightforwardly to

$$K_0(\phi, \phi^*) = \frac{c}{2m} \left\{ (\text{Re}(\phi_+))^2 + \left[\log \left(\frac{|e^{\phi_-}| + \sqrt{|e^{\phi_-}|^2 - 4}}{2} \right) \right]^2 \right\}, \quad (4.13)$$

where $\text{Re} \phi_+/2m$ and $\text{Re} \phi_-/m$ are the center of mass and the relative distance between two walls, respectively, according to Eq.(4.4). This Kähler potential gives precisely the Kähler metric found by D.Tong[10]. Let us note that finding the Kähler potential from the Kähler metric may appear straight-forward, but is often nontrivial in reality. It is one of the merits of our formulation to obtain the Kähler potential directly without going through computation of Kähler metric and its integration.

4.3 Exact Result at a Finite Gauge Coupling $g^2 c = m^2$

For particular discrete finite values of gauge coupling, we have previously found exact solutions [12] of the master equation (2.9). Especially, if we choose

$$g^2 c = m^2, \quad (4.14)$$

and $M = m \text{diag}(2, 1, 0)$, we obtain a double-wall solution for full moduli including the distance between the walls. With the parametrization (4.12), the solution Ω is given by

$$\Omega = |e^{\frac{\phi_+}{2}}|^2 e^{2my} \left(e^{my} |e^{-\frac{\phi_+}{2}}| + e^{-my} |e^{\frac{\phi_+}{2}}| + \sqrt{6 + |e^{\phi_-}|} \right)^2. \quad (4.15)$$

We can evaluate the integral formula (2.59) and (2.60) for the Kähler potential $K(\phi, \phi^*)$ by inserting the solution Ω in Eq.(4.15) and by integrating over y . Using the same method as that in the previous section, we obtain the first term in Eq.(2.60) explicitly as

$$c \int_{-\infty}^{\infty} dy \log \Omega = \frac{2c}{m} \left\{ \frac{1}{4} (\text{Re}(\phi_+))^2 + \left[\log \left(\frac{\sqrt{6 + |e^{\phi_-}|} + \sqrt{2 + |e^{\phi_-}|}}{2} \right) \right]^2 \right\}. \quad (4.16)$$

The third term in Eq.(2.60) is found to be

$$\frac{1}{2g^2} \int_{-\infty}^{\infty} dy (\Omega^{-1} \partial_y \Omega)^2 = -\frac{4m}{g^2} \sqrt{\frac{6 + |e^{\phi_-}|}{2 + |e^{\phi_-}|}} \log \left(\frac{\sqrt{6 + |e^{\phi_-}|} + \sqrt{2 + |e^{\phi_-}|}}{2} \right), \quad (4.17)$$

where we used an integration formula

$$\int_{-L}^L dz \frac{e^{2z}}{(e^z + x_1)(e^z + x_2)} = L + \frac{x_1 \log x_1 - x_2 \log x_2}{x_1 - x_2} + \mathcal{O}(e^{-L}), \quad (4.18)$$

whereas the contribution from the second term in Eq.(2.60) vanishes. Since the gauge coupling g^2 is related to the mass parameter m by Eq.(4.14), the Kähler potential of the exact solution at this finite value of the gauge coupling is given by

$$K(\phi, \phi^*) = \frac{2c}{m} \left\{ \frac{1}{4} (\text{Re}(\phi_+))^2 + \left(\log \left(\frac{\sqrt{6 + |e^{\phi_-}|} + \sqrt{2 + |e^{\phi_-}|}}{2} \right) \right)^2 - 2 \sqrt{\frac{6 + |e^{\phi_-}|}{2 + |e^{\phi_-}|}} \log \left(\frac{\sqrt{6 + |e^{\phi_-}|} + \sqrt{2 + |e^{\phi_-}|}}{2} \right) \right\}. \quad (4.19)$$

This Kähler potential gives correctly the Kähler metric found in Ref.[12]. Again it is the merit of our formulation to obtain the Kähler potential directly.

4.4 Boojums as a Solution of Double-wall Effective Lagrangian

As an application of the effective Lagrangian on the domain wall, we can consider 1/2 BPS lump (semi-local vortex) on the double wall configuration. The resulting configuration is of course a 1/4

BPS lumps stretching between two walls. Let us emphasize that we have an effective Lagrangian of the double-wall system, rather than an effective Lagrangian on a single wall. Therefore we can consider lumps stretched between two walls, in contrast to the previous studies of lumps on a single domain wall such as studies of “BIon” [57]. The 1/2 BPS equation can be derived from the wall effective Lagrangian with the Kähler potential (4.19), and is given by

$$\bar{\partial}\phi_- = 0, \quad (4.20)$$

where $z = x^1 + ix^2$ is a complex variable for the two spacial dimensions of the lump profile. The energy density of the 1/2 BPS state of the effective theory is given by

$$\mathcal{E} = \frac{\partial^2 K(\phi, \phi^*)}{\partial \phi^i \partial \phi^{j*}} \partial_\mu \phi^i \partial^\mu \phi^{j*} \Big|_{\text{BPS}} = 2\partial\bar{\partial}K(\phi, \phi^*)|_{\text{BPS}} = \frac{1}{2}\partial_{(2)}^2 K(\phi, \phi^*)|_{\text{BPS}}, \quad (4.21)$$

where $\partial_{(2)}^2$ is the two-dimensional Laplacian on the x^1, x^2 plane. A lump located at the origin $z = 0$ with the vorticity k and the size (and phase) z_0 is given by

$$e^{\frac{\phi_-}{2}} = (z/z_0)^k, \quad e^{\phi_+} = \text{const.} \quad (4.22)$$

The lump acts as a magnetic charge at $z = 0$ on the wall, and the flux escapes to infinity on the wall. This produces a logarithmic bending of the wall and the excitation energy diverges at large $r \equiv |z|$. Conversely the logarithmically bent wall can be regarded as a lump whose cross section becomes bigger as $y \rightarrow \pm\infty$. Therefore we introduce a cut-off at the radius $r = \Lambda$ to evaluate the energy precisely. Defining $|z_0| = r_0$, the energy of the lump inside the radius Λ is found to be

$$\begin{aligned} E_k(\Lambda) &= \frac{1}{2} \int_{r=0}^{r=\Lambda} dx^2 \partial_{(2)}^2 K = \pi \left[r K' \right]_{r=0}^{r=\Lambda} \\ &= \frac{4\pi c k}{m} \left\{ k \log \frac{\Lambda}{r_0} - 1 + \mathcal{O} \left(\left(\frac{r_0}{\Lambda} \right)^{2k} \right) \right\} \\ &= 2\pi c k L_k \left(\frac{\Lambda}{r_0} \right) - \frac{4\pi m k}{g^2} + \mathcal{O} \left(\left(\frac{r_0}{\Lambda} \right)^{2k} \right). \end{aligned} \quad (4.23)$$

This energy diverges in the limit of $\Lambda \rightarrow \infty$. The divergence gives precisely the energy of the semi-local vortex with length $L_k \left(\frac{\Lambda}{r_0} \right) = \frac{2k}{m} \log \frac{\Lambda}{r_0}$ stretched between two walls, since the energy density of the vortex per unit length is given by $2\pi c k$. Moreover, we find a finite contribution $-4\pi m k/g^2$. This contribution comes from the third term of the Kähler potential in Eq.(2.60). By using the relation $g^2 c = m^2$, the sign and magnitude of the contribution is found to agree precisely with the contribution from a monopole in the Higgs phase for the $U(1)$ case which is called boojum [37, 38, 39]. Please note that we have obtained the negative contribution from the boojum correctly, in contrast to the fact that the monopoles in non-Abelian gauge theory have positive energy as usual [58].

5 Discussion

In the superfield formalism, we have obtained the Kähler potential density of the effective actions on BPS domain walls (2.60) and vortices (3.22) in non-Abelian gauge theory. We need the

explicit solutions Ω to the master equation (2.46) or (3.19), in order to obtain explicit expression of Kähler potentials. The applicable range is the same with the component formalism. However our method have many advantages to the component formalism. The most remarkable merit is that the Kähler potential can be directly obtained with little effort due to manifest supersymmetry. This is in contrast to the component formalism, in which one has to integrate the Kähler metric twice to obtain the Kähler potential. One might consider this is just a technical advantage, but it is not the case. It needed tedious calculation to obtain the Kähler potentials (and even Kähler metric), even for the Abelian case [28]. Using our method we have been able to obtain the Kähler potentials very easily even for non-Abelian case, which is a new result. Another merit is that we do not have to guess multiplet structure of supersymmetry, unlike the case of the component formalism, in which identification of fermionic superpartners sometimes brings trouble or difficulty.

Extension to composite solitons is an interesting problem. For instance loops in domain wall webs [60] or vortices stretched between walls [37] can have localized modes because they do not change boundary conditions. Extension to higher derivative terms is also one of future directions. Higher derivative corrections to translational zero modes should sum up to the form of the Nambu-Goto Lagrangian if we include infinite number of derivatives using nonlinear realizations (see for instance [61]). However higher derivative corrections to orientational zero modes are not known in general. The only exception is a forth order term of orientational zero modes of domain walls found in a model with degenerate masses [62], which turns out to be the Skyrme term. Relation between the Manton's method discussed in this paper with nonlinear realizations or mode expansions should be clarified.

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A Effective Lagrangian in terms of Component Fields

Assuming $H^2 = 0$, we vary the fundamental Lagrangian to obtain the equations of motion (with $H^2 = 0$) as

$$\begin{aligned}
0 &= \mathcal{D}^M \mathcal{D}_M H^1 + \frac{g^2}{2} (c - H^1 H^{1\dagger}) H^1 - \Sigma^2 H^1 + 2\Sigma H^1 M - H^1 M^2, \\
0 &= \mathcal{D}^M \mathcal{D}_M \Sigma - \frac{g^2}{2} (\{\Sigma, H^1 H^{1\dagger}\} - 2H^1 M H^{1\dagger}), \\
0 &= \frac{1}{g^2} \mathcal{D}^N F_{NM} - \frac{i}{g^2} [\Sigma, \mathcal{D}_M \Sigma] - \frac{i}{2} (H^1 \mathcal{D}_M H^{1\dagger} - \mathcal{D}_M H^1 H^{1\dagger}).
\end{aligned} \tag{A.1}$$

Using the order assignments (2.12), (2.13) and (2.14), we can expand the field equations in powers of λ . The lowest order equations are of order λ^0 , and are automatically satisfied by the BPS equations (2.3) and (2.4). By taking the order λ part of the equations of motion, we obtain the field equation (2.15) for the fluctuation W_μ . A solution of this equation is given by

$$W_\mu = i \left((\delta_\mu S^\dagger) S^{\dagger-1} - S^{-1} (\delta_\mu^\dagger S) \right), \quad (\text{A.2})$$

leading to $\mathcal{D}_\mu S^{-1} = -S^{-1} (\delta_\mu \Omega) \Omega^{-1}$. Without the help of the unbroken supersymmetry, it is not at all straightforward to find out this solution, contrary to the procedure in (2.51)-(2.56) where the unbroken supersymmetry has facilitated to obtain the solution dramatically. Uniqueness of the solution (A.2) comes from the uniqueness of the solution of the master equation (2.9). We can confirm that the solution (A.2) leads to the following two equations and thus satisfies Eq.(2.15)

$$\mathcal{D}_\mu H^1 = S^\dagger \delta_\mu (\Omega^{-1} H_0) e^{My}, \quad (\text{A.3})$$

$$\mathcal{D}_\mu \Sigma + i F_{\mu y}(W) = S^\dagger \delta_\mu (\Omega^{-1} \partial_y \Omega) S^{\dagger-1}, \quad (\text{A.4})$$

where we use an identity

$$\Omega^{-1} \delta_2 (\delta_1 \Omega \Omega^{-1}) = \delta_1 (\Omega^{-1} \delta_2 \Omega) \Omega^{-1}, \quad (\text{A.5})$$

and another relation resulting from the master equation (2.9)

$$\delta \partial_y (\Omega^{-1} \partial_y \Omega) = -g^2 \delta \left(\Omega^{-1} H_0 e^{2My} H_0^\dagger \right). \quad (\text{A.6})$$

The correct gauge transformation of the solution of the gauge field is guaranteed by that of S : $S \rightarrow S' = SU$ with $U = U(y, \phi(x), \phi^*(x))$

$$W_\mu \rightarrow W'_\mu = U^{-1} W_\mu U - i U^{-1} \partial_\mu U, \quad U^\dagger = U^{-1}. \quad (\text{A.7})$$

Let us obtain the effective Lagrangian $\mathcal{L}^{\text{eff}} \equiv \int dy \mathcal{L}$ by substituting Eqs.(A.3) and (A.4) to the fundamental Lagrangian in five-dimensions and by integrating over the extra dimension y

$$\begin{aligned} \mathcal{L}^{\text{eff}} + T_w &= \int dy \text{Tr} \left[\mathcal{D}^\mu H \mathcal{D}_\mu H^\dagger + \frac{1}{g^2} (\mathcal{D}^\mu \Sigma - i F^\mu_y(W)) (\mathcal{D}_\mu \Sigma + i F_{\mu y}(W)) \right] \\ &= \int dy \text{Tr} \left[\Omega \delta^\mu (\Omega^{-1} H_0) e^{2My} \delta_\mu^\dagger (H_0^\dagger \Omega^{-1}) + \frac{1}{g^2} \Omega \delta^\mu (\Omega^{-1} \partial_y \Omega) \Omega^{-1} \delta_\mu^\dagger (\partial_y \Omega \Omega^{-1}) \right] \\ &= \int dy \text{ReTr} \left[c \delta^\mu (\Omega^{-1} \delta_\mu^\dagger \Omega_0) + \frac{\partial_y^2}{2g^2} ((\delta^\mu \Omega) \Omega^{-1} (\delta_\mu^\dagger \Omega) \Omega^{-1}) \right] \\ &= \int dy \delta^\mu \delta_\mu^\dagger \left[\left(c - \frac{\partial_y^2}{g^2} \right) \log \det \Omega + \frac{1}{2g^2} \text{Tr} (\Omega^{-1} \partial_y \Omega)^2 \right] - \frac{1}{g^2} \text{ReTr} [\Omega^{-1} \partial_y \Omega \delta^\mu (\Omega^{-1} \delta_\mu^\dagger \Omega)] \Big|_{-\infty}^{\infty} \\ &\equiv \delta^\mu \delta_\mu^\dagger K(\phi, \phi^*) = K_{ij}(\phi, \phi^*) \partial^\mu \phi^i \partial_\mu \phi^{j*}, \end{aligned} \quad (\text{A.8})$$

which gives the Kähler potential K as

$$K(\phi, \phi^*) = \int dy \left[c \log \det \Omega + \frac{1}{2g^2} \text{Tr} (\Omega^{-1} \partial_y \Omega)^2 \right]. \quad (\text{A.9})$$

Let us list some useful formulas. The definition of S in Eq.(2.6) gives

$$(\mathcal{D}_y + \Sigma) S^{-1} = 0 \quad \rightarrow \quad \Sigma = \frac{1}{2} S^{-1} \partial_y \Omega S^{\dagger-1}. \quad (\text{A.10})$$

If we consider $[\mathcal{D}_y, \mathcal{D}_y + \Sigma] S^{-1}$, we obtain

$$\mathcal{D}_y \Sigma = \frac{1}{2} S^{-1} \partial_y [(\partial_y \Omega) \Omega^{-1}] S. \quad (\text{A.11})$$

B Variation of Superfield Form of 6D Lagrangian

In this appendix we summarize useful formulas for the variation of vector superfields, especially the variation due to gauge transformations. Let us first evaluate the exponential of vector superfield

$$\begin{aligned}\delta e^{zV} &= \int_0^1 dt e^{ztV} z \delta V e^{z(1-t)V} = \left(\frac{e^{zL_V} - 1}{L_V} \times \delta V \right) e^{zV} \\ &= e^{zV} \left(\frac{1 - e^{-zL_V}}{L_V} \times \delta V \right) = e^{zV} \left(\frac{1}{L_V} \times \delta V \right) - \left(\frac{1}{L_V} \times \delta V \right) e^{zV},\end{aligned}\quad (\text{B.1})$$

where the operation L_V is defined in Eq.(3.9).

We need to introduce the Wess-Zumino-Witten like term (3.8) in order to achieve gauge invariance under the complexified $U(N_C)$ gauge transformations. Let us denote a term without Φ , c or \mathbf{H}^i in the $\int d^4\theta$ terms as \mathcal{L}_1

$$\mathcal{L}_1 \equiv \int d^4\theta \text{Tr} \left[\frac{2}{g^2} e^{-2\mathbf{V}} \bar{\partial} e^{2\mathbf{V}} e^{-2\mathbf{V}} \partial e^{2\mathbf{V}} \right] = \int d^4\theta \text{Tr} \left[\frac{4}{g^2} \bar{\partial} \mathbf{V} \frac{\cosh(2L_{\mathbf{V}}) - 1}{L_{\mathbf{V}}^2} \partial \mathbf{V} \right]. \quad (\text{B.2})$$

We can combine this term with the Wess-Zumino-Witten like term and obtain

$$\begin{aligned}\mathcal{L}_2 \equiv \mathcal{L}_1 + \mathcal{L}_{\text{WZW}} &= \int d^4\theta \text{Tr} \left[\frac{4}{g^2} \bar{\partial} \mathbf{V} \frac{e^{2L_{\mathbf{V}}} - 1 - 2L_{\mathbf{V}}}{L_{\mathbf{V}}^2} \partial \mathbf{V} \right] \\ &= \int d^4\theta \int_0^1 dx \int_0^x dy \text{Tr} \left[\frac{16}{g^2} \bar{\partial} \mathbf{V} e^{2yL_{\mathbf{V}}} \partial \mathbf{V} \right].\end{aligned}\quad (\text{B.3})$$

Thus we obtain the fundamental Lagrangian (3.12) after combining with the Wess-Zumino-Witten like term.

To obtain the variation of the fundamental Lagrangian (3.12), it is useful to vary the above term (B.3)

$$\begin{aligned}\delta \mathcal{L}_2 &= - \int_0^1 dx \int_0^x dy \text{Tr} \left[\frac{16}{g^2} \delta \mathbf{V} \right. \\ &\quad \times \left. \left(\bar{\partial}(e^{2yL_{\mathbf{V}}} \partial \mathbf{V}) + \partial(e^{-2yL_{\mathbf{V}}} \bar{\partial} \mathbf{V}) + \frac{e^{2yL_{\mathbf{V}}} - 1}{L_{\mathbf{V}}} [(e^{-2yL_{\mathbf{V}}} \bar{\partial} \mathbf{V}), \partial \mathbf{V}] \right) \right].\end{aligned}\quad (\text{B.4})$$

We can use the following formulas to simplify the variation

$$\int_0^x dy \left(\bar{\partial}(e^{2yL_{\mathbf{V}}} \partial \mathbf{V}) + \partial(e^{-2yL_{\mathbf{V}}} \bar{\partial} \mathbf{V}) \right) = \bar{\partial} \left(\frac{e^{2xL_{\mathbf{V}}} - 1}{2L_{\mathbf{V}}} \partial \mathbf{V} \right) + \partial \left(\frac{1 - e^{-2xL_{\mathbf{V}}}}{2L_{\mathbf{V}}} \bar{\partial} \mathbf{V} \right), \quad (\text{B.5})$$

$$\begin{aligned}\int_0^x dy \frac{e^{2yL_{\mathbf{V}}} - 1}{L_{\mathbf{V}}} [(e^{-2yL_{\mathbf{V}}} \bar{\partial} \mathbf{V}), \partial \mathbf{V}] &= \int_0^x dy \frac{1}{L_{\mathbf{V}}} \{ [\bar{\partial} \mathbf{V}, e^{2yL_{\mathbf{V}}} \partial \mathbf{V}] + [\partial \mathbf{V}, e^{-2yL_{\mathbf{V}}} \bar{\partial} \mathbf{V}] \} \\ &= \frac{1}{L_{\mathbf{V}}} \left\{ \left[\bar{\partial} \mathbf{V}, \frac{e^{2xL_{\mathbf{V}}} - 1}{2L_{\mathbf{V}}} \partial \mathbf{V} \right] + \left[\partial \mathbf{V}, \frac{1 - e^{-2xL_{\mathbf{V}}}}{2L_{\mathbf{V}}} \bar{\partial} \mathbf{V} \right] \right\} \\ &= -\bar{\partial} \left(\frac{1}{2L_{\mathbf{V}}} \right) [(e^{2xL_{\mathbf{V}}} - 1) \partial \mathbf{V}] - \partial \left(\frac{1}{2L_{\mathbf{V}}} \right) [(1 - e^{-2xL_{\mathbf{V}}}) \bar{\partial} \mathbf{V}].\end{aligned}\quad (\text{B.6})$$

Thus we obtain the variation of the term \mathcal{L}_2

$$\begin{aligned}
\delta\mathcal{L}_2 &= 4\delta \int_0^1 dx \int_0^x dy \text{Tr} [\bar{\partial}\mathbf{V} e^{2yL_{\mathbf{V}}} \partial\mathbf{V}] \\
&= -2 \int_0^1 dx \text{Tr} \left[\delta\mathbf{V} \frac{1}{L_{\mathbf{V}}} \{ \bar{\partial} (e^{2xL_{\mathbf{V}}} \partial\mathbf{V}) - \partial (e^{-2xL_{\mathbf{V}}} \bar{\partial}\mathbf{V}) \} \right] \\
&= -\text{Tr} \left[\delta\mathbf{V} \frac{1}{L_{\mathbf{V}}} \{ \bar{\partial} ((\partial e^{2\mathbf{V}}) e^{-2\mathbf{V}}) - \partial (e^{-2\mathbf{V}} \bar{\partial} e^{2\mathbf{V}}) \} \right] \\
&= -\text{Tr} \left[\delta\mathbf{V} \frac{e^{2L_{\mathbf{V}}} - 1}{L_{\mathbf{V}}} \partial (e^{-2\mathbf{V}} \bar{\partial} e^{2\mathbf{V}}) \right] = -\text{Tr} [e^{-2\mathbf{V}} (\delta e^{2\mathbf{V}}) \partial (e^{-2\mathbf{V}} \bar{\partial} e^{2\mathbf{V}})] . \tag{B.7}
\end{aligned}$$

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